Dynamics of a Sliding Particle in a Rotating Drum

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The motion of a sliding particle, influenced by friction, in a rotating drum is investigated. A differential equation is formulated for general friction laws. Assuming a constant coefficient of friction, the equation is exactly solvable. For a velocity dependent coefficient of friction, perturbation methods may be used. The nonperturbed system is solved and with the help of the *averaging method*, the perturbed system can be examined for periodic motions. Different friction laws lead to qualitatively different behaviours, including a stable fixed point in the phase plot, marginally stable orbits and stable limit cycle behaviour. A friction law proposed by E. Rabinowicz reproduces the major result of the experiment, i.e., convergence towards a limit cycle. Other laws will be discussed.

1 Introduction

Friction has an important influence on the behaviour of granular matter. To obtain a better understanding of friction, we have investigated the motion of a single sliding particle in a rotating drum. A fascinating feature of this setup is that small changes of the friction law lead to different behaviours of the particle.

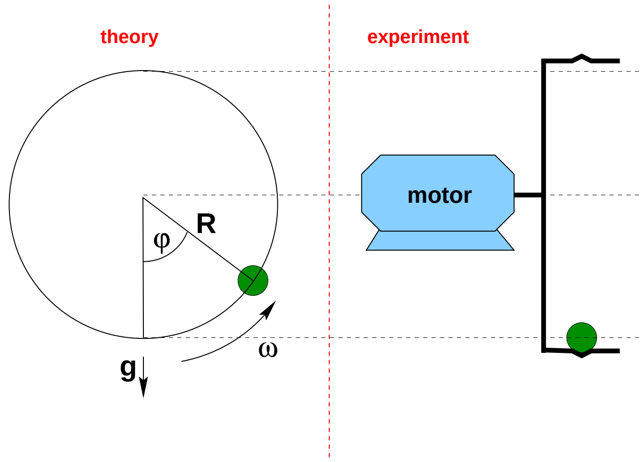


Figure 1: Experimental setup and theoretical simplification (left) of the drum

The experimental setup assures that particle movement is restricted to one-dimensional excursions, hence the position can be described by $\varphi(t)$. Let the angular velocity of the drum be ω, its radius R. Since the friction force F_{fric} is proportional to the normal force F_{norm}, the differential equation for this problem is, if we assume the friction coefficient to be a function of the relative velocity $v_{rel} = R \cdot (\omega - \dot{\varphi}) = R\dot{\varphi}_{rel}(t)$:

$$R\ddot{\varphi}(t) = -g\sin\varphi(t) + \mu(\dot{\varphi}_{rel}(t)) \cdot \underbrace{\left\{g\cos\varphi(t) + R\dot{\varphi}^2(t)\right\}}_{F_{norm}}. \tag{1}$$

For *constant* μ ($\mu = \mu_0$), this equation can be integrated once, which yields

$$\dot{\varphi}^2 = -2g\frac{1}{1+4\mu_0^2} \cdot \left\{(2\mu_0^2 - 1)\cos\varphi - 3\mu_0\sin\varphi\right\} + 2e^{2\mu_0\varphi} \cdot c. \tag{2}$$

With the averaging method,[2,4] we assume that the presence of the perturbation will change the constant of integration c into a slowly varying function of time:

$$c = c_0(\tau) + \varepsilon c_1(t, \tau) + \cdots \tag{3}$$

where $\tau = \varepsilon t$ is a slow time. Differentiating once yields

$$\dot{c} = \varepsilon(c_{0\tau}(\tau) + c_{1t}(t, \tau)) + \mathcal{O}(\varepsilon^2) \tag{4}$$

Integrating over one period, we eliminate c_1 and using the time derivative of equation (2), we obtain

$$\varepsilon c_{0\tau} = -\frac{2}{T_p}\int_{\varphi^{min}}^{\varphi^{max}} d\varphi\ \dot{\varphi}e^{-2\mu_0\varphi}\dot{\varphi} \cdot \varepsilon\left[(\mu(\dot{\varphi}) - \mu_0) \cdot \left(\frac{g}{R}\cos\varphi + \dot{\varphi}^2\right)\right] \tag{5}$$

where T_p is the periodicity. Of course, T_p itself depends on c_0 via equation (2).

$$T_p = 2\int_{\varphi^{min}}^{\varphi^{max}} d\varphi\ \frac{1}{\dot{\varphi}}.$$

If equation (5) happens to have a fixed point c_0^*, there is a periodic solution of the perturbed equation, the periodicity of which is given by $T_p(c_0^*)$.

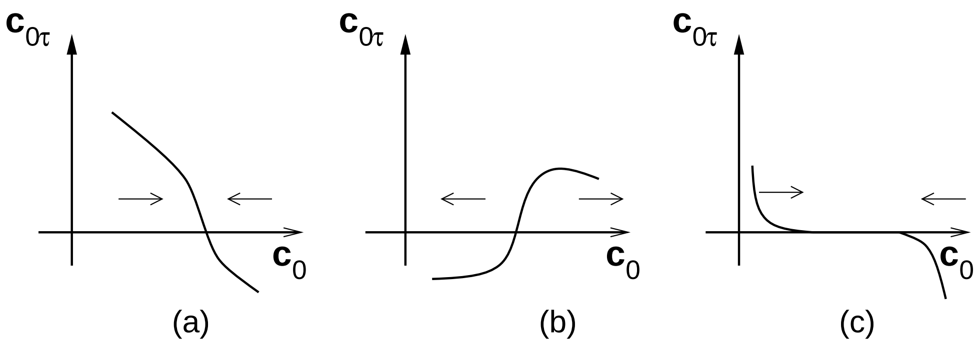


Figure 2: Different types of fixed points

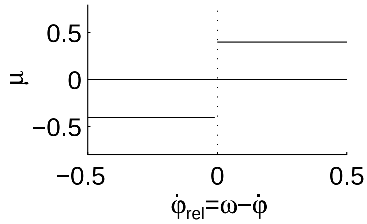
Examining figure 2, we can see three different types of fixed points. In case (a), we have a stable orbit, case (b) is unstable. In case (c), where $c_{0\tau} = 0$ on a whole interval, we have a continuum of marginally stable orbits.

2 Friction Laws

2.1 Coulomb's law without static friction

In this case, the friction coefficient depends only on the direction of the velocity vector (Coulomb's law).

$$\mu(\dot{\varphi}_{rel}) = \begin{cases} \mu_{kin} & \text{if} \quad \dot{\varphi}_{rel} \geq 0 \\ -\mu_{kin} & \text{if} \quad \dot{\varphi}_{rel} < 0 \end{cases} .$$



We observe two different kinds of trajectories: For small starting values $\varphi(0)$ and $\dot{\varphi}(0) = 0$, the particle's velocity will be small and $\dot{\varphi}_{rel}$ is always > 0. Hence, we have the unperturbed solution, the trajectory is marginally stable and $c_{0\tau} \equiv 0$.

For larger values of $\varphi(0)$ and $\dot{\varphi}(0) = 0$, starting conditions lie outside the periodic orbit and $\dot{\varphi}_{rel}$ can become < 0. As a result, the particle can dissipate energy, the trajectories are approaching the periodic orbit.

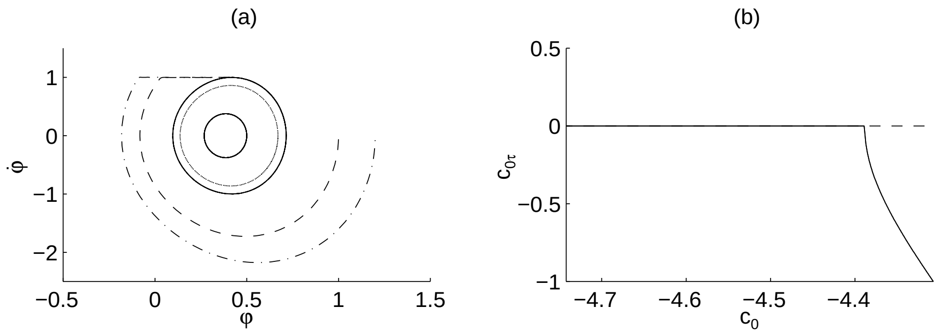
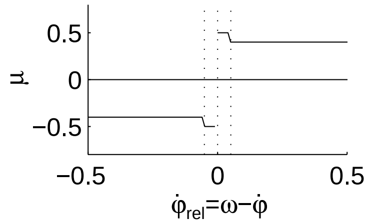


Figure 3: (a) shows a phase plot $\omega = 1$, $\mu = 0.4$ R $= 1$ and $\varphi(0) = 0.5, 0.7, 1.0, 1.2$; (b) shows a typical plot for equation (5) for Coulomb's law. The parameters are the same as in (a).

2.2 Coulomb's law with static friction

This is similar to the previous version of Coulombs's law but incorporates μ_{stat} within a small range of velocity to simulate static friction.

$$\mu(\dot{\varphi}_{rel}) = \begin{cases} \mu_{kin} & \text{if} \quad \dot{\varphi}_{rel} \geq \dot{\varphi}_0 \\ \mu_{stat} & \text{if} \quad 0 \leq \dot{\varphi}_{rel} < \dot{\varphi}_0 \\ -\mu_{stat} & \text{if} \quad -\dot{\varphi}_0 \leq \dot{\varphi}_{rel} < 0 \\ -\mu_{kin} & \text{if} \quad \dot{\varphi}_{rel} < -\dot{\varphi}_0 \end{cases}$$



Here we can obtain four different kind of trajectories in the phase plot. As initial condition on $\dot{\varphi}$, we always take $\dot{\varphi}(0) = 0$.

For small $\varphi(0)$, the trajectory is marginally stable, this is again the unperturbed solution. In a small range of $\varphi(0)$ values, the trajectory can reach the area of static friction. The particle gains energy and approaches the stable state. Then we have the stable periodic orbit itself, we can see the corresponding root of equation 5 in figure 4b. According to figure 2, we know that the orbit is stable for the rightmost fixed point. For large values of $\varphi(0)$, the trajectories approach the stable orbit, similar to the case of the Coulomb's law without static friction.

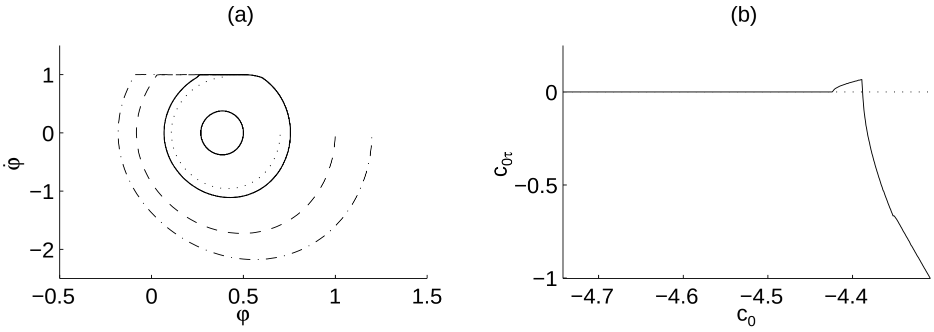
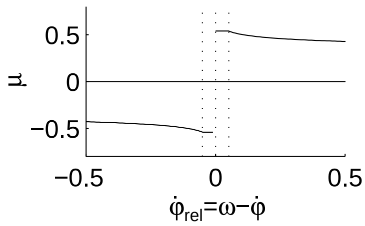


Figure 4: (a) shows a phase plot $\omega = 1$, $\mu = 0.4$ R = 1 and $\varphi(0) = 0.5, 0.7, 1.0, 1.2$; (b) shows a typical plot for equation (5) if we have Coulomb's law with static friction. The parameters are the same as in (a).

2.3 Friction law as suggested by Rabinowicz

This case is similar to the last case, but we have a velocity dependence of $\mu \propto v^{-0.1}$. This friction law was proposed by E. Rabinowicz.[1]

$$\mu(\dot{\varphi}_{rel}) = \begin{cases} \mu_{kin}\dot{\varphi}_{rel}^{-0.1} & \text{if} \quad \dot{\varphi}_{rel} \geq \dot{\varphi}_0 \\ \mu_{kin}\mathrm{v}_0^{-0.1} & \text{if} \quad 0 \leq \dot{\varphi}_{rel} < \dot{\varphi}_0 \\ -\mu_{kin}\mathrm{v}_0^{-0.1} & \text{if} \quad -\dot{\varphi}_0 \leq \dot{\varphi}_{rel} < 0 \\ -\mu_{kin}\dot{\varphi}_{rel}^{-0.1} & \text{if} \quad \dot{\varphi}_{rel} < -\dot{\varphi}_0 \end{cases}$$



Here we have three different kinds of trajectories. For $\varphi(0) < \varphi(0)_{\text{periodic}}$ (and $\dot{\varphi}(0) = 0$), the particle can gain energy and approach the periodic state. Then there is the periodic orbit itself, as one can see in plot 5b. For $\varphi(0) > \varphi(0)_{\text{periodic}}$, the trajectories dissipate energy and approach the stable orbit.

Comparing this behaviour with experimental results,[3] we obtain good agreement regarding the dependence of the oscillation frequency of the particle on the rotation frequency of the drum. Nonidealities of the latter may lead to locking effects whenever the particle frequency approaches an integer multiple of the drum frequency, and these effects can be accounted for by simple modifications of the basic differential equation (1), simulating "bumps" in the drum or off-center rotation.

On the other hand, Coulomb's law with or without static friction is not able to reproduce the typical behaviour of the experiment. The reason is that with Coulomb's law, the fixed point of (1), given by $\tan(\varphi) = \mu_0$, is elliptic, hence structurally unstable and destructible by arbitrarily small perturbations. Since Coulomb's law is only an approximation to real life, these small perturbations must always be expected to be present. And they are, as experiments show.

Finally, we should like to mention that friction laws with monotously increasing velocity dependence of the friction coefficient lead to nonoscillatory behaviour, with the motion of the particle converging to a fixed point. This again is different from what is observed in experiments.

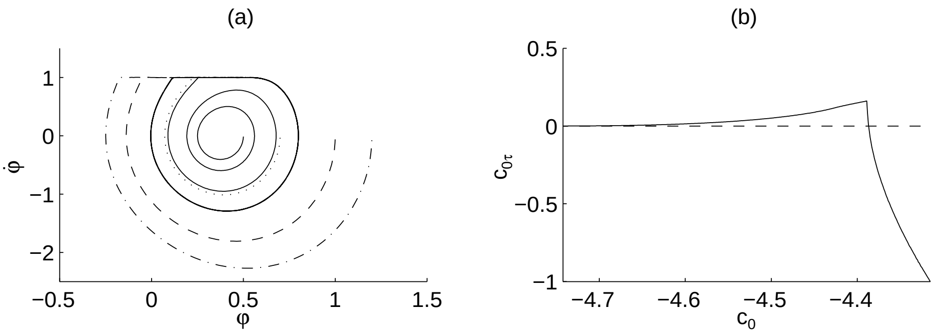


Figure 5: (a) shows a phase plot $\omega = 1$, $\mu = 0.4$ R = 1 and $\varphi(0) = 0.5, 0.7, 1.0, 1.2$ (b) shows a typical plot for equation (5) if we have the friction law suggested by Rabinowicz. The parameters are the same as in (a).

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References

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4. S. Linz, Eur. J. Phys **16**, 67-72 (1995).